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# On a Class of Singular Boundary Value Problems Which Contains the Boundary Conditions $X'(0) = X(1) = 0$*

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## 0. INTRODUCTION

In this paper we prove an existence theorem for the boundary value problem

$$x'' + f(t, x) = 0 \tag{0.1}$$

$$x(0) = 0, \qquad \lambda x(1) + \mu x'(1) = 0, \tag{0.2}$$

where $\lambda, \mu \geqslant 0$, $\lambda + \mu > 0$ and $f: (0, 1) \times (0, \infty) \to (0, \infty)$ is a continuous function which does not have, in general, a continuous extension to $[0, 1] \times [0, \infty)$. In this case, we say that problem (0.1) is singular.

By the change of variables $y(t) = x(1-t)$, we get a parallel result for the boundary conditions

$$\lambda x(0) - \mu x'(0) = 0, \qquad x(1) = 0, \tag{0.3}$$

and we apply this to proving some existence theorems for the following boundary value problem:

$$x'' + ct^{-1}x' + f(t, x) = 0, \qquad x'(0) = x(1) = 0. \tag{0.4}$$

See Theorems 0.4–0.5 below.

These problems appear in a variety of applications. See [2, 5, 6], where $f(t, x)$ has the form $\alpha(t)x^{-p}$, for some $p > 0$ and some continuous function $\alpha: (0, 1) \to (0, \infty)$. See also [1, 3, 4, 7] for more general $f$. Problem (0.4) also arises in the search for positive radially symmetric solutions to the problem $[\Delta u + f(|x|, u) = 0,\ x \in \Omega,\ u \mid \Gamma = 0]$, where $\Omega$ is the open unit ball

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in $\mathbb{R}^{c+1}$ (if $c$ is an integer) centered at the origin, $\Gamma$ is its boundary, and $|x|$ is the Euclidean norm of $x$. See [3, 4].

In the following we write $E_1 = C^2(0, 1) \cap C^1(0, 1] \cap [0, 1]$ and $E_0 = C^2(0, 1) \cap C^1[0, 1) \cap C[0, 1]$. We shall prove:

0.1. THEOREM. *Suppose that there are continuous functions* $\alpha, \gamma: (0, 1) \to (0, \infty)$ *and a decreasing function* $g: (0, M) \to (0, \infty)$ (*some* $0 < M \leqslant \infty$); *such that*

$$\gamma(t) \leqslant f(t, x) \leqslant \alpha(t)\, g(x) \qquad \text{in} \quad (0, 1) \times (0, M). \tag{0.5}$$

*Assume further that* $t\alpha(t)$ *is integrable over* $[0, 1]$ *and*

$$\int_0^M dx/g(x) > \int_0^1 t\alpha(t)\, dt. \tag{0.6}$$

*If* $\mu > 0$, *then* (0.1)–(0.2) *has a solution* $u$ *in* $E_1$ *such that* $u(t) \leqslant M$ *in* $[0, 1]$.

*Remark.* If $M = \infty$ in (0.5), then (0.6) is automatically true. Moreover, in this case there is $\infty > N > 0$ such that (0.6) is true if we replace $M$ by $N$. So, we can assume that $M$ is finite.

From the change of variables $y(t) = x(1 - t)$, we get:

0.2. THEOREM. *Assume* (0.5) *and* $\mu > 0$. *If* $(1 - t)\,\alpha(t)$ *is integrable over* $[0, 1]$ *and*

$$\int_0^M dx/g(x) > \int_0^1 (1 - t)\,\alpha(t)\, dt, \tag{0.7}$$

*then* (0.1), (0.3) *has a solution* $u$ *in* $E_0$ *such that* $u(t) \leqslant M$.

0.3. *Remarks.* (a) If $\mu = 0$ we can prove that (0.1)–(0.2) has one solution in $C^2(0, 1) \cap C[0, 1]$ if $t(1 - t)\,\alpha(t)$ is integrable over $[0, 1]$. This result appears in [7].

(b) Assume that $\alpha(t)\, g(kt)$ is integrable over $[0, 1]$ for all $k > 0$. Then, $t\alpha(t)$ is integrable over $[0, 1]$ and each solution $u$ of (0.1)$-$(0.2) belongs to $C^1[0, 1]$. (Compare with Theorem 2.2 of [3].)

*Proof.* Since $g(x)/x$ is decreasing in $(0, \infty)$ then $g(x) \geqslant g(1)x$ in $(0, 1)$ and thus, $t\alpha(t)$ is integrable over $[0, 1]$. On the other hand, it is easy to prove that there exists $k > 0$, such that $u(t) \geqslant kt$, in $[0, 1]$. From this,

$$|u'(t) - u'(s)| \leqslant \int_s^t \alpha(\xi)\, g(k\xi)\, d\xi$$

for $0<s<t<1$, and then $u'$ is uniformly continuous on $(0, 1)$. Consequently, $u'$ has a continuous extension to $[0, 1]$, and the proof follows easily.

We have a parallel remark for Theorem 0.2.

(c) Let $f(t, x)=[x^2+(a-t)^2]x^{-3}$, where $a=(2\lambda+\mu)^{-1}(\lambda+\mu)$. Then, $f(t, x)$ is decreasing in $x>0$ and $f(t, x)\leqslant K(x^{-1}+x^{-3})$ for some constant $K>0$. Thus, $f$ satisfies the hypotheses of theorem 0.1. In this case, the unique solution to (0.1)–(0.2) is given by $u(t)=(2at-t^2)^{1/2}$. (See Corollary 2.3 below.) Note that $u\notin C^1[0, 1]$.

EXAMPLE. *Let $f(t, x)=\alpha(t)\,\phi(x)x^{-p}$; where $p>0$ and $\alpha$: $(0, 1)\to(0, \infty)$, $\phi$: $[0, \infty)\to(0, \infty)$ are continuous functions such that $t\alpha(t)$ is integrable over $[0, 1]$ and $\phi$ is increasing. Further, if $\phi(x)\to\infty$ as $x\to\infty$, assume that $\phi$ is differentiable on $(0, \infty)$ and $\phi'(x)/x^p\to 0$ as $x\to\infty$. Then, $f(t, x)$ satisfies the hypotheses in theorem* 0.1. (*This includes the "model case"*, $\phi\equiv 1$.)

*Proof.* Let $I$ denote the integral of $t\alpha(t)$ over $[0, 1]$, and let us consider the following cases:

(1) $\phi$ is bounded. In this case let us choose $K$, $M>0$ such that $\phi(x)\leqslant K$ for $x>0$ and

$$M^{p+1}>(p+1)KI. \tag{0.8}$$

(2) $\phi$ is unbounded. In this case, by L'Hopital's rule,

$$\lim_{x\to\infty}(p+1)\phi(x)/x^{p+1}=\lim_{x\to\infty}\phi'(x)/x^p=0,$$

and so there exists $M>0$ such that

$$(p+1)\phi(M)I<M^{p+1}. \tag{0.9}$$

Now, let us define $g(x)=\phi(M)x^{-p}$ for $0<x<M$ and $\gamma(t)=\phi(0)\,\alpha(t)/M^p$. Then, $g$ is decreasing; (0.5) is satisfied and (0.6) is equivalent to (0.8) (resp. (0.9)) if $\phi$ is bounded (resp. unbounded). So, the proof is complete.

Theorem 0.1 is related to some results in [1, 3, 7]. In fact, our proof take some ideas from [1, 7].

Using Theorem 0.2 and a comparison result (Theorem 2.1 below), we shall prove the following:

0.4. THEOREM. *Let* $C \in C^0[0,1] \cap C^1(0,1)$ *be an increasing function, which is positive in* $(0, 1]$. *If the hypotheses of theorem* 0.2 *are satisfied, then the problem*

$$x'' + \frac{C'(t)}{C(t)} x' + f(t, x) = 0 \tag{0.10}$$

$$x'(0) = x(1) = 0 \tag{0.11}$$

*has a solution* $u$ *in* $E_0$ *such that* $u(t) \leqslant M$.

*Remarks.* (a) If $\alpha(t)\, g(k(1-t))$ is integrable over $[0, 1]$ for all $k>0$ then, each solution $u$ of (0.10)–(0.11) belongs to $C^1[0, 1]$.

*Proof.* We remark that $(Cu')' + Cf(\cdot, u) \equiv 0$ in $(0, 1)$ and then, $Cu'$ is uniformly continuous in $(0, 1)$. See Remark 0.3b. Thus, $Cu'$ has a continuous extension $v$ to $[0, 1]$ and hence, $v/C$ is a continuous extension of $u'$ to $[1/2, 1]$. The proof follows easily.

(b) Theorem 0.4 was considered in [3, Th. 4.1], where it was assumed that $C(t) = t^n$ and $f(t, x) = \alpha(t) x^{-p}$, for some $n \geqslant 1$ and $p>0$. Note that $C(t) = t^c$ satisfies the hypotheses in Theorem 0.4, for every $c>0$. Moreover, in this case, (0.10)–(0.11) reduces to (0.4).

(c) We first prove that Theorem 0.4 is valid, when $C(0)>0$. Thus, the problem $[x'' + C_n'(t)x'/C_n(t) + f(t, x) = 0,\ x'(0) = x(1) = 0]$ has one solution $u_n$; where $C_n(t) = C(1/n + (1-1/n)t)$. The proof will follow, from a suitable modification of the arguments in Theorem 0.1.

If (0.5) fails, we have the following alternative result:

0.5. THEOREM. *Let* $C$ *be as in theorem* 0.4, *and suppose that* $f(t, x)$ *is decreasing in* $x$. If $(1-t)f(t, k(1-t))$ *is integrable over* $[0, 1]$ *for all* $k>0$ *then,* (0.10)–(0.11) *has a unique solution* $u$ *in* $E_0$. *Moreover,* $(1-t)\,u'(t) \to 0$ *as* $t \to 1$. *If in addition,* $f(t, k(1-t))$ *is integrable over* $[0, 1]$ *for all* $k>0$, *then* $u \in C^1[0, 1]$.

*Remark.* The existence part of Theorem 0.5 was proved in [4] under the additional assumptions that

(1) $f(t, x) \to \infty$ as $x \to 0$ uniformly on compact subsets of $(0, 1)$,

(2) $f(t, x) \to 0$ as $x \to \infty$ uniformly on compact subsets of $(0, 1)$,

(3) $f$ has a continuous extension to $[0, 1) \times (0, \infty)$, while the uniqueness part needs

(4) $f(t, x)$ is strictly decreasing in $x$ and locally Lipschitz.

To be complete, we prove a uniqueness result (Theorem 3.4 below), which contains the corresponding results in [1, 3, 4, 7]. From this corollary we get that the problem (0.1)–(0.2) (resp. (0.1)–(0.3), (0.10)–(0.11)) has at most one solution if $f(t, x)$ is decreasing in $x$.

⟨1⟩ In this section we shall prove theorem 0.1. We need several notations and intermediate results.

Let $J$ be an interval of $\mathbb{R}$. We denote by $C(J)$ (resp. $C^n(J)$, $n \geqslant 1$ integer) the space of all continuous (resp. $n$-times continuously differentiable) functions $u\ J\colon \to \mathbb{R}$, endowed, if $J$ is compact, with the norm $\|u\|_0 = \sup\{|u(t)|\colon t \in J\}$ (resp. $\|u\|_n = \max\{\|u^{(i)}\|_0\colon\ 0 \leqslant i \leqslant n\}$). If $J = [0, 1]$, we write $C(J) = C^0$ and $C^n(J) = C^n$.

1.1. Proposition. *Let $u \in C^2$ satisfying the boundary conditions* (0.2) *and assume that there are $\alpha \in C^0$ and $g\colon [0, M] \to (0, \infty)$, some $M > 0$, such that $\alpha, g > 0$, $g$ is decreasing, $0 \leqslant u(t) \leqslant M$ and*

$$-u''(t) \leqslant \alpha(t)\, g(u(t)) \tag{1.1}$$

*in* $[0, 1]$. *If $s \in (0, 1]$ denotes the unique critical point of $u$ then*

$$\int_0^{u(s)} dx/g(x) \leqslant \int_0^s t\alpha(t)\, dr. \tag{1.2}$$

*Proof.* See [7, Prop. 1.1].

1.2. Theorem. *Let $f_0\colon [0, 1] \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$ be continuous and assume that there are $\alpha, g$ as in proposition* 1.1 *such that $0 < f_0(t, x, y) \leqslant \alpha(t)\, g(x)$ in $(0, 1) \times (0, M)$. If*

$$\int_0^M dx/g(x) > \int_0^1 t\alpha(t)\, dt, \tag{1.3}$$

*then the problem*

$$x'' + f_0(t, x, x') = 0, \qquad x(0) = 0, \qquad \lambda x(1) + \mu x(1) = 0$$

*has a solution $u$ in $C^2$ such that $\|u\|_0 \leqslant M$.*

*Proof.* It follows from the same arguments as in Th. 1.2 of [7].

1.3. Proposition. *Let $\alpha \in C(0, 1)$ be positive. If $t\alpha(t)$ is integrable over* $[0, 1]$, *then there is a sequence $(a_n)$ in $(0, 1)$ such that $a_n \to 0$ and*

$$a_n \int_{a_n}^1 \alpha(t)\, dt \to 0.$$

*Proof.* See [7, Prop. 1.3].

In the following, we assume that $\mu > 0$.

1.4. PROPOSITION. *Let $\gamma \in C^0$ be positive in* $(0, 1)$ *and for each integer $n \geqslant 3$, let us define $\gamma_n \in C[0,1]$, by $\gamma_n(t) = (1-2/n)^2 \gamma(1/n + (1-2/n)t)$. If $w_n \in C^2$ is the only solution to the problem: $x'' + \gamma_n(t) = 0$, $x(0) = \lambda x(1) + \mu x'(1) = 0$, then there is $\varepsilon > 0$, such that $w_n(t) \geqslant \varepsilon t$ in* $[0, 1]$ *for $n \geqslant 3$.*

*Proof.* It follows from the same arguments as in Prop. 1.4 of [7].

*Proof of Theorem* 0.1. We can assume that $M < \infty$. Now, let us define $T_n(t) = 1/n + (1-2/n)t$; $\alpha_n(t) = (1-2/n)^2 \alpha(T_n(t))$, $\gamma_n(t) = (1-2/n)^2 \gamma(T_n(t))$, and

$$f_n(t, x) = (1 - 1/2n)^2 f(T_n(t), x + 1/n),$$

for all integers $n \geqslant 3$ and $t$ in $[0, 1]$.

It is clear that

$$\gamma_n(t) \leqslant f_n(t, x) \leqslant \alpha_n(t)\, g(x + 1/n) \leqslant \alpha_n(t)\, g(x) \tag{1.4}$$

for $0 \leqslant x \leqslant M - 1/n$. We also have that

$$\int_0^{M-1/n} dx/g(x+1/n) = \int_{1/n}^{M} dx/g(x) \to \int_0^M dx/g(x)$$

$$\int_0^1 s\alpha_n(s)\, ds = \int_{1/n}^{1-1/n} (s - 1/n)\, \alpha(s)\, ds$$

$$< \int_{1/n}^{1-1/n} s\alpha(s)\, ds \to \int_0^1 s\alpha(s)\, ds.$$

Since (0.6) holds, we can assume that

$$\int_0^{M-1/n} dx/g(x+1/n) > \int_0^1 s\alpha_n(s)\, ds \qquad \text{for} \quad n \geqslant 3$$

and by Theorem 1.2, the problem

$$x'' + f_n(t, x) = 0, \qquad x(0) = \lambda x(1) + \mu x'(1) = 0 \tag{1.5}$$

has a solution $u_n$ such that

$$\|u_n\| \leqslant M - 1/n < M \qquad \text{for} \quad n \geqslant 3. \tag{1.6}$$

Let $t_n \in (0, 1]$ be the unique critical point of $u_n$. By Proposition 1.1, we get that

$$\int_0^{u_n(t_n)} dx/g(x) \leqslant \int_0^{T_n(t_n)} s\alpha(s)\, ds. \tag{1.7}$$

On the other hand, by the arguments in Thm. 1.5 of [7], we find $k>0$ such that

$$u_n(t) \geqslant kt \text{ in } [0, 1] \qquad \text{for} \quad n>2. \tag{1.8}$$

Since $t_n$ gives the global maximum of $u_n$ on $[0, 1]$, then (1.8) implies that $u_n(t_n) \geqslant u_n(1) \geqslant k$, for all $n \geqslant 3$, and from (1.7) we have $\inf\{t_n\} > 0$.

CLAIM. *Let $J=[\varepsilon, 1]$, where $0<\varepsilon<\inf\{t_n\}$ and $\varepsilon<1/2$. Then, there is a subsequence of $\{u_n\}$ which converges in $C^1(J)$ to a solution $w \in C^2(J)$ of* (0.1). *Moreover, $w$ has a critical point.*

*Proof.* By (1.8) there is $\delta>0$ such that $u_n(t) \geqslant \delta$ in $[\varepsilon, 1]$ and by (1.4)–(1.5),

$$-u_n''(t) \leqslant C\alpha_n(t) \qquad \text{in} \quad [\varepsilon, 1]$$

where $C := g(\delta)$. From this,

$$|u_n'(t)-u_n'(s)| \leqslant C \int_s^t \alpha_n(\tau)\, d\tau \leqslant C(1-2/n) \int_{T_n(s)}^{T_n(t)} \alpha(s)\, ds$$

for $\varepsilon \leqslant s<t \leqslant 1$; then $\{u_n'\}$ is equicontinuous in $[\varepsilon, 1]$, since $\alpha$ is integrable over $[\varepsilon, 1]$. Note that $T_n(\varepsilon) \geqslant \varepsilon$.

Moreover, for $s=t_n$ (0r $t=t_n$), we get that

$$|u_n'(t)| \leqslant C \int_\varepsilon^1 \alpha(s)\, ds, \qquad t \in J.$$

Thus, by Ascoli's theorem, we can assume that there exists $w \in C^1(J)$ such that $u_n \to w$ in $C^1(J)$. By (1.5) and (1.8), we have that $w \in C^2(J)$ and $w$ is a solution to (0.1).

On the other hand, we can assume without loss of generality that $t_n \to t_0 > \varepsilon$. So, $u_n'(t_n) \to u'(t_0)$ and the proof of the claim is complete.

Now, it is not hard to prove that there is a solution $u$ of (0.1) in $C^2(0, 1]$, which has a critical point in $(0, 1]$, such that for each compact interval $J$ of $(0, 1]$, there exists a subsequence of $\{u_n\}$ which converges to $u$ in $C^1(J)$. In particular, $\lambda u(1)+\mu u'(1)=0$.

From (1.4)–(1.5), we get that

$$-u_n'(t)/g(u_n(t)) \leqslant \int_t^{t_n} \alpha_n(s)\,ds$$

for $0 < t \leqslant t_n$, and by integration in $[0, t] \subseteq (0, t_n]$ one has

$$\int_0^{u_n(t)} dx/g(x) \leqslant t \int_t^1 \alpha_n(s)\,ds + \int_0^t s\alpha_n(s)\,ds.$$

Consequently,

$$\int_0^{u(t)} dx/g(x) \leqslant t \int_t^1 \alpha(s)\,ds + \int_0^t s\alpha(s)\,ds,$$

and then $u(a_n) \to 0$, where $\{a_n\}$ is the sequence given by Proposition 1.3.

Let $s > 0$ be the unique critical point of $u$. Then, $u$ is increasing in $(0, s)$. (Remember that $u'' < 0$) and so, $u(t) \to 0$ as $t \to 0^+$.) Thus, the proof is complete.

⟨2⟩ In this section we shall prove theorem 0.4. We shall use the arguments in Corollary 1.6 of [7] to prove:

2.1. PROPOSITION. *Theorem* 0.4 *is true if* $C(0) > 0$.

*Proof.* Let us define $\psi, \phi \in C^1[0, 1] \cap C^2(0, 1)$, by

$$\psi(t) = \int_0^t C(s)^{-1} dt, \qquad \phi(t) = \psi(t)/\psi(1).$$

Then, $\phi(0) = 0$, $\phi(1) = 1$, $\phi' > 0$ in $[0, 1]$, and $C\phi'' + C'\phi' \equiv 0$ in $(0, 1)$.

Now, let us define $f_0(t, x) = \phi'(\phi^{-1}(t))^{-2} f(\phi^{-1}(t), x)$ and $\alpha_0(t) = \phi'(\phi^{-1}(t))^{-2}\alpha(\phi^{-1}(t))$. Then, $f_0(t, x) \leqslant \alpha_0(t) g(x)$, for $0 < x < M$ and

$$\begin{aligned}\int_0^1 (1-t)\,\alpha_0(t)\,dt &= \int_0^1 (1-\phi(s))\,\phi'(s)^{-1}\alpha(s)\,ds \\ &= \int_0^1 C(s)\left(\int_s^1 C(t)^{-1} dt\right)\alpha(s)\,ds \\ &\leqslant \int_0^1 (1-s)\,\alpha(s)\,ds,\end{aligned}$$

since $C(s) \leqslant C(t)$ if $s \leqslant t$. From this, and Theorem 0.2, the problem

$$x'' + f_0(t, x) = 0, \qquad x'(0) = x(1) = 0$$

has one solution $v$ in $E_0$, such that $v(t) \leqslant M$. Now, it is easy to verify that $u(t) := v(\phi(t))$ is a solution to our problem.

*Remark.* Let $\phi \in C(0, 1)$ be positive. It is easy to prove that there is $\psi \in C[0, 1]$, such that $0 < \psi \leqslant \phi$ in (0.1). Thus, we can assume that the function $\gamma$ in (0.5), belongs to $C[0, 1]$.

*Proof of Theorem* 0.4. Let us write $T_n(t) = 1/n + (1 - 1/n)t$ and $C_n(t) = C(T_n(t))$ for all integers $n > 1$ and $t \in [0, 1]$. By Proposition 2.1, the problem

$$\begin{aligned} x'' + [C_n'(t)x'/C_n(t)] + f(t, x) = 0 \\ x'(0) = 0, \qquad x(1) = 0 \end{aligned} \tag{2.1}$$

has one solution $u_n$ such that $u_n(t) \leqslant M$. Note that

$$(C_n u_n')' + C_n f(t, u_n) = 0 \qquad \text{in} \quad (0, 1), \tag{2,2}$$

and hence

$$-C_n(\tau)\, u_n'(\tau) = \int_0^\tau C_n(\xi) f(\xi, u_n(\xi))\, d\xi. \tag{2.3}$$

In particular, $u_n' < 0$ in $(0, 1)$ and thus, $u_n$ is strictly decreasing in $[0, 1]$. From this,

$$-u_n'(\tau)/g(u_n(\tau)) \leqslant C_n(\tau)^{-1} \int_0^\tau C_n(\xi)\, \alpha(\xi)\, d\xi \tag{2.4}$$

for $0 \leqslant \tau \leqslant 1$. (Notice that $1/g(x)$ has an extension to $[0, \infty)$ which is continuous at $x = 0$.)

CLAIM 1. *There exists $k > 0$ such that $u_n(t) \geqslant k(1 - t)$ in $[0, 1]$, for all $n \geqslant 1$.*

*Proof.* We can assume that the function $\gamma$ in (0.5) belongs to $C^0$. So, for each integer $n \geqslant 1$, we can define

$$w_n(t) = \int_t^1 C_n(\tau)^{-1} \left( \int_0^\tau C_n(\xi)\, \gamma(\xi)\, d\xi \right) d\tau$$

It is easy to prove that $w_n$ is the unique solution to

$$x'' + C_n'(t)x'/C_n(t) + \gamma(t) = 0, \qquad x'(0) = x(1) = 0.$$

On the other hand,

$$-C_n(t)\, w_n'(t) = \int_0^t C_n(\xi)\, \gamma(\xi)\, d\xi \geqslant k_0 := \int_0^{1/2} C(\xi)\, \gamma(\xi)\, d\xi$$

in $[1/2, 1]$ and, thus, $0 < -w_n'(t) \leqslant 2k := k_0/C(1)$ in $[1/2, 1]$. (Note that $C_n(t) \geqslant C(t)$, since $T_n(t) \geqslant t$.) Hence, $w_n(t) \geqslant 2k(1-t)$ in this interval. But $w_n$ is decreasing and so, $w_n(t) \geqslant w(1/2) \geqslant k$, in $[0, 1/2]$. From this, $w_n(t) \geqslant k(1-t)$ in $[0, 1]$.

Finally, $(C_n w_n')' = -C_n \gamma \geqslant -C_n f(t, u_n) = (C_n u_n')'$ and then, $C_n(t)\, w_n'(t) \geqslant C_n(t)\, u_n'(t)$ since $w_n'(0) = u_n'(0) = 0$. Consequently $u_n'(t) \leqslant w_n'(t)$, and thus $u_n(t) \geqslant w_n(t)$, because $w_n(1) = u_n(1) = 0$. Therefore, the proof of the claim is complete.

CLAIM 2. *For each compact interval* $J := [a, b] \subseteq (0, 1)$, *there exists a subsequence of* $\{u_n\}$, *which converges in* $C^2(J)$, *to a solution* $w \in C^2(J)$ *of* (0.10).

*Proof.* From (2.3) and Claim 1, we have that

$$-u_n'(t) \leqslant g(k(1-b)) \int_0^t \alpha(s)\, ds \qquad \text{in } J$$

since $C_n(\xi) \leqslant C_n(\tau)$ if $\xi \leqslant \tau$; and $f(t, u_n(t)) \leqslant \alpha(t)\, g(u_n(t))$.

Thus, $\{u_n\}$ is bounded in $C^1(J)$ and by (2.1) $\{u_n\}$ is bounded in $C^2(J)$, since $\{C_n'/C_n\}$ is bounded in $C(J)$. Then, we can assume that there is $v$ in $C^1(J)$ such that $u_n \to v$ in $C^1(J)$, and by (2.1),

$$u_n'' \to -C'v'/C - f(\cdot, v) \qquad \text{in } C(J).$$

The proof of Claim 2 follows easily.

Using Claim 2, we can prove the existence of a solution $u$ to (0.10) in $C^2(0, 1)$, with the following property: For each compact interval $J \subseteq (0, 1)$, there exists a subsequence of $\{u_n\}$ which converges to $u$ in $C^2(J)$. Therefore, $u$ is decreasing and then, $u$ has a continuous extension to $[0, 1]$, which we still denote by $u$. Note that $0 < u(t) \leqslant M$ in $(0, 1)$.

Let us define

$$I_1(t) = \int_t^1 (1-\xi)\, \alpha(\xi)\, d\xi, \qquad I_2(t) = (1-t) \int_0^t \alpha(\xi)\, d\xi.$$

Integration of (2.4) in $[t, 1]$ and Fubini's theorem yield

$$\begin{aligned}
\int_0^{u_n(t)} dx/g(x) &\leqslant \int_0^t \left( \int_t^1 C_n(\tau)^{-1} d\tau \right) C_n(\xi)\, \alpha(\xi)\, d\xi \\
&\quad + \int_t^1 \left( \int_\xi^1 C_n(\tau)^{-1}\, d\tau \right) C_n(\xi)\, \alpha(\xi)\, d\xi \\
&\leqslant I_2(t) + I_1(t),
\end{aligned}$$

and then

$$\int_0^{u(t)} dx/g(x) \leqslant I_1(t) + I_2(t) \tag{2.5}$$

for all $t$ in $(0, 1)$.

CLAIM 3. *There exists a sequence $\{a_n\}$ in $[1/2, 1)$ such that $a_n \to 1$ and $I_2(a_n) \to 0$.*

*Proof.* Assume that there exists $c > 0$ such that

$$\int_0^t \alpha(\xi)\, d\xi \geqslant c/(1-t) \qquad \text{in} \quad [1/2, 1].$$

Integrating this relation in $[1/2, r]$, for $1/2 < r < 1$, we get

$$\int_{1/2}^r (r-\xi)\, \alpha(\xi)\, d\xi \to +\infty \qquad \text{as} \quad r \to 1^{-1}.$$

But this is a contradiction, since $(r-\xi)\, \alpha(\xi) \leqslant (1-\xi)\, \alpha(\xi)$. So, the proof of Claim 3 is complete.

Let $\{a_n\}$ be as above. By (2.5) we get $u(a_n) \to 0$, and thus, $u(1) = 0$. On the other hand, by (2.3) and Claim 1, we gave that

$$0 < -u_n'(t) \leqslant g(k(1-t)) \int_0^t \alpha(s)\, ds \qquad \text{if} \quad t \in (0, 1)$$

From here, and Claim 2, $0 \leqslant -u'(t) \leqslant g(k(1-t)) \int_0^t \alpha(s)\, ds$ and so $u'(t) \to 0$ as $t \to 0^+$. Thus, $u \in E_0$ and theproof is complete.

⟨3⟩ In this section we prove a comparison theorem, which will be basic in the proof of Theorem 0.5.

3.1. THEOREM. *Suppose that there exist $u, v \in E_0$ such that*

$$u'' + f(t, u) \geqslant v'' + f(t, v) \qquad \textit{in} \quad (0, 1), \tag{3.1}$$

$$\lambda u(0) - \mu u'(0) \leqslant \lambda v(0) - \mu v(0); \qquad u(1) \leqslant v(1) \tag{3.2}$$

*for some $\lambda, \mu \geqslant 0$; $\lambda + \mu > 0$. If $f(t, x)$ is decreasing in $x$ for each fixed $t \in (0,1)$, then $u \leqslant v$.*

*Proof.* Let us define $\phi(t) = f(t, v(t)) - f(t, u(t))$ and $w = u - v$. Since (3.1) holds and $f(t, x)$ is decreasing in $x$, we have that

$$w'' \geqslant \phi, \; \phi . w \geqslant 0 \qquad \text{in} \quad (0, 1), \tag{3.3}$$

and by (3.2),

$$\lambda w(0) - \mu w'(0) \leqslant 0; \qquad w(1) \leqslant 0. \tag{3.4}$$

Let us fix $\tau \in [0, 1]$ such that $w(\tau) = \sup(w)$. We must prove that $w(\tau) \leqslant 0$. To this end, assume $w(\tau) > 0$ then, $\tau < 1$ and:

CLAIM. $w'(\tau) = 0$.

*Proof.* If $\tau > 0$ there is nothing to prove. Suppose now that $\tau = 0$. Then, $w'(0) \leqslant 0$, since $w(0) = \sup(w)$. From (3.4), $0 \leqslant \lambda w(0) \leqslant \mu w'(0) \leqslant 0$ and so, $\lambda = 0$ and $\mu w'(0) = 0$. Thus, the proof of the Claim is complete.

Let $\xi$ be the point in $(\tau, 1]$ satisfying $w(\xi) = 0$ and $w > 0$ in $(\tau, \xi)$. By (3.3), $w'' \geqslant 0$ in this interval and hence, the same holds for $w'$, since $w'(\tau) = 0$. From this, $w(\xi) \geqslant w(\tau) = 0$, and this contradiction ends the proof.

As an immediate consequence of Theorem 3.1, we obtain the following uniqueness result, which is related to Theorems 3 and 3.1 of [1] and [3], respectively.

3.2. COROLLARY. *If $f(t, x)$ is decreasing with respect to $x$, for each fixed $t$ in* (0, 1), *then the problem* (0.1), (0.3) *has at most one solution. A parallel result holds for the problem* (0.1)–(0.2).

3.3. COROLLARY. *Let $u \in E_0$ be a solution to* (0.10)–(0.11) *and let $v \in E_0$ be a solution to*

$$x'' + f(t, x) = 0, \qquad x'(0) = x(1) = 0. \tag{3.5}$$

*If $f(t, x)$ is decreasing with respect to $x$, and $C$ is as in Theorem* 0.4, *then $u \leqslant v$.*

*Proof.* From the relation $(Cu')' = Cf(t, u) = -Cf(t, u) < 0$ in $(0, 1)$, we obtain $u' < 0$ in $(0, 1)$, since $u'(0) = 0$. So, $u'' + f(u, t) \geqslant u'' + C'u'/C + f(t, u) = 0 = v'' + f(t, v)$, and the proof follows from Theorem 3.1.

3.4. THEOREM. *Let $f$, $C$ be as in Corollary* 3.3. *Then, the problem* (0.10)–(0.11) *has at most one solution.*

*Proof.* Let $u$, $v$ be solutions to (0.10)–(0.11) and write $w = u - v$. Then, $(Cw')'(t) = \phi(t) := C(t)[f(t, v(t)) - f(t, u(t))]$; $\phi \cdot w \geqslant 0$ and $w'(0) = w(1) = 0$.
Now, let us fix $\tau \in [0, 1]$ such that $|w(\tau)| = \sup(|w|)$ and assume $w \not\equiv 0$. Without loss of generality $(w \to -w)$, we can suppose that $w(\tau) > 0$ and, as in the proof of Theorem 3.1, we get a contradiction. Thus, the proof is complete.

⟨4⟩ In this section we shall prove Theorem 0.5. We begin with the following result.

4.1. PROPOSITION. *Let* $h$: $[0, 1] \times [0, \infty) \to (0, \infty)$ *be a continuous function such that* $h(t, x)$ *is decreasing in* $x$. *Then, the problem*

$$x'' + h(t, x) = 0, \qquad x'(0) = x(1) = 0$$

*has a solution.*

*Proof.* Let us define $H$: $[0, 1] \times \mathbb{R} \to \mathbb{R}$ by $H(t, x) = h(t, |x|)$ and notice that $0 < H(t, x) < R := 1 + \sup\{h(t, 0): t \in [0, 1]\}$.

Define $U = \{u \in C^2: \|u\|_2 < R\}$. It is easy to prove that the problem $[x'' + \lambda H(t, x) = 0,\ x'(0) = x(1) = 0]$ has no solution in the boundary of $U$ for any $\lambda$ in $[0, 1]$. From this, the problem $[x'' + H(t, x) = 0,\ x'(0) = x(1) = 0]$ has a solution $u$ in $U$. Since $u'' < 0$ and $u'(0) = u(1) = 0$, then $u > 0$ in $(0, 1)$ and the proof is complete.

In the following we assume that $f(t, x)$ is decreasing in $x$.

4.2. THEOREM. *Assume that there exists* $v, w \in E_0$ *such that* $w(1) = v(1) = 0$ *and*

$$w''(t) + f(t, w(t)) \geqslant 0 \geqslant v''(t) + f(t, v(t)) \qquad in \quad (0, 1).$$

*Assume further that there is a sequence* $\{a_n\}$ *in* $(0, 1)$ *such that* $a_n \to 0$ *and* $w'(a_n) \geqslant 0 \geqslant v'(a_n)$ *for all* $n$. *Then, the problem*

$$x'' + f(t, x) = 0, \qquad x'(0) = x(1) = 0 \tag{4.1}$$

*has a solution* $u \in E_0$.

*Proof.* Note that $w'(0) \geqslant 0 \geqslant v'(0)$ and, by Theorem 3.1, $w \leqslant v$. Now, let us choose a sequence $\{b_n\}$ in $(0, 1)$ such that $a_n < b_n$ and $b_n \to 1$, and define $T_n(t) = a_n + (b_n - a_n)t$, $w_n(t) = w(T_n(t)) - w(b_n)$, $v_n(t) = v(T_n(t)) - w(b_n)$, and $f_n(t, x) = (a_n - b_n)^2 f(T_n(t), x + w(b_n))$, for all integers $n \geqslant 1$ and $t$ in $[0, 1]$.

It is clear that $w_n(1) = 0 \leqslant v_n(1)$, $w_n'(0) \geqslant 0 \geqslant v_n'(0)$, and

$$w_n'' + f_n(t, w_n) \geqslant 0 \geqslant v_n'' + f_n(t, v_n) \qquad \text{in} \quad (0, 1).$$

Let $u_n$ be a solution to the problem

$$x'' + f_n(t, x) = 0, \qquad x'(0) = x(1) = 0. \tag{4.2}$$

See Proposition 4.1. By Theorem 3.1 we have, $w_n \leqslant u_n \leqslant v_n$. In particular, $\{u_n\}$ is bounded in $C^0$.

Let $J \subseteq (0, 1)$ be a compact interval, since $w_n \to w$ in $C(J)$, there exists $\varepsilon > 0$ such that $u_n(t) \geqslant \varepsilon$ in $J$ for all $n > 1$ and by (4.2), $\{u_n''\}$ is bounded in $C(J)$. Consequently, $\{u_n\}$ is bounded in $C^2(J)$ and by Ascoli's Theorem, we can assume that there is $z$ in $C^1(J)$ such that $u_n \to z$ in $C^1(J)$. By (4.2) we get that: $z \in C^2(J)$, $u_n \to z$ in $C^2(J)$ and $z'' + f(t, z) = 0$. Thus, Eq. (0.1) has one solution $u \in C^2(0, 1)$ with the following property: for each compact interval $J \subseteq (0, 1)$ there exists a subsequence of $\{u_n\}$ which converges to $u$ in $C^2(J)$. In particular, $u$ is bounded and decreasing and so, $u$ has a continuous extension to $[0, 1]$, which we still denote by $u$. Note that $w \leqslant u \leqslant v$ and then, $u(1) = 0$.

From the relation $w_n \leqslant u_n \leqslant v_n$ we have $f_n(t, w_n) \geqslant f_n(t, u_n) \geqslant f_n(t, v_n)$ and hence $w_n'' \leqslant u_n'' \leqslant v_n''$. By integration, we have $w_n'(t) - w_n'(0) \leqslant u_n'(t) \leqslant v_n'(t) - v_n'(0)$ in $[0, 1)$ and thus,

$$w'(t) - w'(0) \leqslant u'(t) \leqslant v'(t) - v'(0) \qquad \text{in} \quad (0, 1).$$

Therefore, $u'(t) \to 0$ as $t \to 0$ and the proof is complete.

4.3. Corollary. *Suppose that there is $w \in E_0$ as in Theorem 4.2, and assume that $(1 - t) f(t, w(t))$ is integrable over $[0, 1]$. Then, problem (4.1) has a solution.*

*Proof.* Since $(1 - t) f(t, w(t))$ is integrable over $[0, 1]$, the problem

$$x'' + f(t, w(t)) = 0, \qquad x'(0) = x(1) = 0$$

has exactly one solution $v$. Note that $v'' < 0$ in $(0, 1)$ and then, the same holds for $v'$. In particular, $v'(a_n) \leqslant 0$.

On the other hand, $(w - v)'' \leqslant 0$ and thus, $w \leqslant v$. Hence, $v'' + f(t, v) \leqslant v'' + f(t, w) = 0$; and the proof follows from Theorem 4.2.

4.4. Corollary. *Assume that $(1 - t) f(t, k(1 - t))$ is integrable over $[0, 1]$ for all $k > 0$. Then, (4.1) has a solution.*

*Proof.* By the arguments in Th. 0.4 of [7], there exists $\delta > 0$ such that $f(t, \delta) \geqslant 9\delta$ for $t \in [1/3, 2/3]$. Now, let us define $w \in C^2$ by

$$w(t) = \delta \qquad \text{in} \quad [0, 1/3]$$

$$w(t) = 27\delta(t - 1/3)^4 - 18\delta(t - 1/3)^3 + \delta \qquad \text{in} \quad [1/3, 2/3]$$

$$w(t) = 2\delta(1 - t) \qquad \text{in} \quad [2/3, 1].$$

It is easy to prove that: $w'' \leqslant 0$, $w''<0$ in $(1/3, 2/3)$, $\min(w'') = w''(1/2) = -9\delta$, $w(1)=0$, $w' \equiv 0$ in $[0, 1/3]$, $w'<0$ in $(1/3, 1)$. In particular, $\max(w) = w(0) = \delta$. From this,

$$w''(t)+f(t, w(t)) = f(t, w(t)) > 0 \qquad \text{in} \quad [0, 1/3] \cup [2/3, 1]$$

$$w''(t)+f(t, w(t)) \geqslant -9\delta + f(t, \delta) \geqslant 0 \qquad \text{in} \quad [1/3, 2/3].$$

On the other hand, it is clear that $w(t) \geqslant k(1-t)$ for some $k>0$ and the proof follows from Corollary 4.3, since $f(t, w(t)) \leqslant f(t, k(1-t))$.

*Proof of Theorem* 0.5. We first assume that $C(0)>0$. With the notations of Propositions 2.1, we have

$$\min(\phi')(1-s) \leqslant 1-\phi(s) = \phi(1)-\phi(s) \leqslant \max(\phi')(1-s)$$

$$\int_0^1 1-t)f_0(t, k(1-t))\,dt \leqslant \min(\phi')^{-1} \int_0^1 (1-\phi(s))f(s, k(1-\phi(s)))\,ds$$

and hence $(1-t)f_0(t, k(1-t))$ is integrable over $[0, 1]$ for all $k>0$. The proof, in this case, follows as in Proposition 2.1.

To prove the general case, let us define $C_n(t)$ as in theorem 0.4. Then, problem (2.1) has one solution $u_n$. On the other hand, let $U$ be the unique solution to (4.1). By Corollary 3.3, $u_n \leqslant U$ and so, $\{u_n\}$ is bounded in $C^0$.

Define now $\gamma(t) = f(t, \max(U))$. By Claim 1 of Theorem 0.4, there exists $k>0$ such that $u_n(t) \geqslant k(1-t)$ in $[0, 1]$ for all $n>1$. Thus, by the arguments in Theorem 0.4,

$$-u_n'(t) \leqslant \int_0^t f(s, k(1-s))\,ds \tag{4.3}$$

for all $t \in [0, 1)$ and hence $\{w_n''\}$ is bounded in $C(J)$, for all compact intervals $J \subseteq (0, 1)$. From this, (0.10)–(0.11) has one solution $u \in C^2(0, 1) \cap C[0, 1]$ such that $k(1-t) \leqslant u(t) \leqslant U(t)$. In particular, $u(1)=0$. By (4.3), $u'(t) \to 0$ as $t \to 0^+$, and so, $u \in E_0$.

On the other hand, $((1-t)\,Cu')' = -Cu' - (1-t)f(t,u)$, and by the arguments in Remark 0.3(b), $(1-t)C(t)u'(t)$ has a continuous extension to $[0, 1]$. Thus, the same holds for $(1-t)u'(t)$ and then $(1-t)u'(t) \to m$ for some real number $m \leqslant 0$. (Remember that $u_n' < 0$ in $(0, 1)$.) If $m<0$, then there exists $c>0$ such that $u'(t) \geqslant c/(1-t)$ in $(1/2, 1)$ and hence

$$\int_{1/2}^1 dt/(1-t) \leqslant c^{-1}u(1/2).$$

This contradiction proves that $m=0$.

The rest of the proof follows from the arguments in Remark 0.3(b) and Remark (a) to Theorem 0.4.

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